Assignment 5

This homework is due *Tuesday* Oct 9. Since there is no class on Oct 9, this assignment can be submitted on *Thursday*, Oct 11.

There are total 57 points in this assignment. 47 points is considered 100%. If you go over 47 points, you will get over 100% for this homework and it will count towards your course grade (up to 115%).

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge*. Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers sections 3.1–3.2 in Bartle–Sherbert.

(1) (a) [3pt] (Theorem 3.2.3) Let $X = (x_n)$ and $Y = (y_n)$ be sequences in $\mathbb{R}$ converging to x and y, respectively. Prove that $X - Y$ converges to $x - y$.

(b) [3pt] (Exercise 3.2.3) Show that if X and Y are sequences in $\mathbb{R}$ such that X and $X + Y$ converge, then Y converges.

(c) [3pt] (Exercise 3.2.2b) Give an example of two sequences X, Y in $\mathbb{R}$ such that XY converges, while X and Y do not.

(2) In this exercise you have to deliver specific inequalities from the definition of the convergent sequence. In each case below, find a number $K \in \mathbb{N}$ such that the corresponding inequality holds for all $n > K$. Give a *specific natural number* as your answer, for example $K = 1000$, or $K = 2 \cdot 10^7$, or $K = 139$, etc. (Not necessarily the smallest possible.)

You can (but you are discouraged to) use a calculator if you want to. However, 1) this problem can be done without using a calculator, 2) even if you do use one, your answers still should easily verifiable without one.

(a) [2pt] $\left|\frac{100-n}{n} - (-1)\right| < 0.054352$,

(b) [3pt] $\left|\frac{200^{10}n+10^{100}}{n^2-10^{200}}\right| < 0.1$,

(c) [3pt] $\left|1/3^n - 1/n^2 + 100/n^5\right| < 0.01$,

(d) [3pt] $\left|\frac{\cos(863n)}{\log n}\right| < 0.032432$,

(e) [4pt] (See example 3.1.11(c)) $|\sqrt[n]{n} - 1| < 0.1$,

(3) Reminder. Recall that a sequence $X = (x_n)$ in $\mathbb{R}$ **does not** converge to $x \in \mathbb{R}$ if there is an $\varepsilon_0 > 0$ such that for any $K \in \mathbb{N}$ there is $n_0 > K$ such that following inequality holds: $|x - x_n| \geq \varepsilon_0$.

In each case below find a *real number* ε_0 that demonstrates that (x_n) does not converge to x.

(a) [2pt] $x_n = 1 + 0.1 \cdot (-1)^{n+1}$, $x = 1$,

(b) [2pt] $x_n = 1/n$, $x = 1/17$,

(c) [2pt] $x_n = (-1)^n n^2$, $x = 0$.

— *see next page* —

(4) REMINDER. Recall definition of a sequence in $\mathbb{R}$ converging to an $x \in \mathbb{R}$: Let $X = (x_n)$ be a sequence in $\mathbb{R}$, let $x \in \mathbb{R}$. $X = (x_n)$ converges to x if $\forall \varepsilon > 0\ \exists K \in \mathbb{N}\ \forall n > K, |x - x_n| < \varepsilon$.

Below you can find (erroneous!) "definitions" of a sequence converging to x. In each case describe, exactly which sequences are "converging to x" according to that "definition".

(a) [3pt] Let $X = (x_n)$ be a sequence in $\mathbb{R}$, let $x \in \mathbb{R}$. $X = (x_n)$ "converges to x" if $\forall \varepsilon > 0\ \forall K \in \mathbb{N}\ \forall n > K, |x - x_n| < \varepsilon$.
(*If you are confused at this point, think of the problem this way: suppose for some sequence (x_n) and a number $x \in \mathbb{R}$ you know that statement (a) is true. What can you say about about (x_n)?*)

(b) [3pt] Let $X = (x_n)$ be a sequence in $\mathbb{R}$, let $x \in \mathbb{R}$. $X = (x_n)$ "converges to x" if $\exists K \in \mathbb{N}\ \forall \varepsilon > 0\ \forall n > K, |x - x_n| < \varepsilon$.

(c) [3pt] Let $X = (x_n)$ be a sequence in $\mathbb{R}$, let $x \in \mathbb{R}$. $X = (x_n)$ converges to x if $\exists \varepsilon > 0\ \exists K \in \mathbb{N}\ \forall n > K, |x - x_n| < \varepsilon$.

(d) [4pt] Let $X = (x_n)$ be a sequence in $\mathbb{R}$, let $x \in \mathbb{R}$. $X = (x_n)$ "converges to x" if $\forall \varepsilon > 0\ \exists K \in \mathbb{N}\ \exists n > K, |x - x_n| < \varepsilon$.

(5) (Exercise 3.1.8) Let (x_n) be a sequence in $\mathbb{R}$, let $x \in \mathbb{R}$.

(a) [4pt] Prove that $\lim(x_n) = 0$ if and only if $\lim(|x_n|) = 0$.

(b) [3pt] Prove that if (x_n) converges to x then $(|x_n|)$ converges to $|x|$.

(c) [3pt] Give an example to show that the convergence of $(|x_n|)$ does not imply the convergence of (x_n).

(6) [4pt] (Exercise 3.2.7) If (b_n) is a bounded sequence and $\lim(a_n) = 0$, show that $\lim(a_n b_n) = 0$. Explain why Theorem 3.2.3 (Arithmetic properties of limit, "$\lim XY = \lim X \cdot \lim Y$") *cannot* be used.